MATH 2010B Advanced Calculus I Lecture Notes (Week 2) Martin Li

<u>Last time</u> $\langle \, , \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

<u>Cross Product in $\mathbb{R}^3$</u>

Define "$\times$" : $\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\vec{a} \times \vec{b} := \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$

if $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc$$

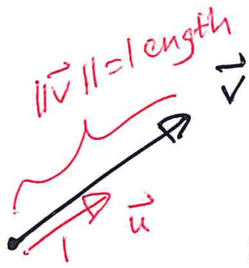
<u>Recall:</u> (det. of 3×3 matrix)

$$\det \begin{pmatrix} \boxed{a} & \boxed{b} & \boxed{c} \\ d & e & f \\ g & h & i \end{pmatrix} := a \begin{vmatrix} e & f \\ h & i \end{vmatrix} + b \begin{vmatrix} f & d \\ i & g \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\therefore \quad \vec{a} \times \vec{b} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

where $\{\vec{i}, \vec{j}, \vec{k}\} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ standard basis

<u>Recall</u>: Any nonzero vector $\vec{v}$ in $\mathbb{R}^n$ can be described by its "magnitude" (a positive number) and its "direction" (a unit vector in $\mathbb{R}^n$).



$$\vec{v} = \underbrace{\|\vec{v}\|}_{\text{magnitude}} \cdot \underbrace{\left(\frac{\vec{v}}{\|\vec{v}\|} \right)}_{\text{direction}} \qquad \text{where } \vec{v} \neq \vec{0}.$$

<u>E.g.</u> $\vec{v} = (1,1,1)$ $\qquad \|\vec{v}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$.

$$\vec{v} = \sqrt{3} \cdot \frac{\vec{v}}{\sqrt{3}} = \sqrt{3} \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

Geometric Definition of $\vec{a} \times \vec{b}$

Thm: If $\vec{0} \neq \vec{a}, \vec{b} \in \mathbb{R}^3$, then $\vec{a} \times \vec{b}$ is the unique vector in $\mathbb{R}^3$ s.t. :

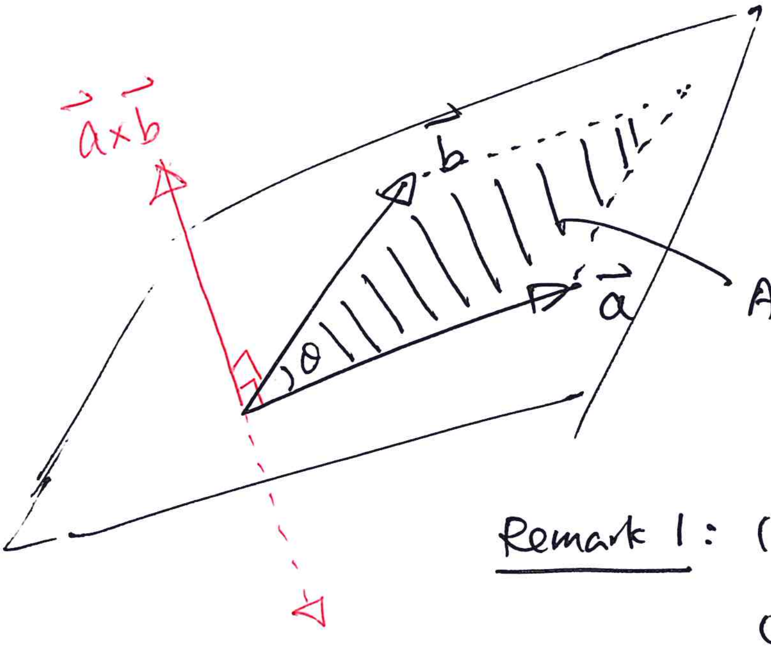
(1) $\vec{a} \times \vec{b} \perp \vec{a}$ and $\vec{a} \times \vec{b} \perp \vec{b}$.

(2) $\{\vec{a}, \vec{b}, \vec{a} \times \vec{b}\}$ gives the standard orientation.

(if $\vec{a} \not\parallel \vec{b}$)

(3) $\|\vec{a} \times \vec{b}\|$ = area of parallelogram spanned by $\vec{a}$ and $\vec{b}$

$$= \|\vec{a}\| \, \|\vec{b}\| \, |\sin\theta|$$



$\{\vec{a}, \vec{b}, \vec{a} \times \vec{b}\}$ right hand rule.

Area $= \|\vec{a} \times \vec{b}\|$.

Remark 1: $\left.\begin{array}{l} (1) + (2) \Rightarrow \text{direction} \\ (3) \Rightarrow \text{length} \end{array}\right\} \Rightarrow$ unique.

Remark: 2 : • $\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined for all n.

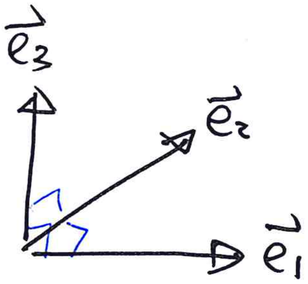
• $\times : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ only for $n = 3$.

Ex: We can "define a cross product" for general dimension using geometric definition.

$$\text{"}\times\text{"} : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n-1} \longrightarrow \mathbb{R}^n$$

(see textbook).

Examples



$$\begin{cases} \vec{e_1} \times \vec{e_2} = \vec{e_3} \\ \vec{e_1} \times \vec{e_3} = -\vec{e_2} \end{cases}$$

$$\vec{e_3} \times \vec{e_1} = \vec{e_2} = -\vec{e_1} \times \vec{e_3}.$$

Fact: $\boxed{\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}}$

Triple Products (consider $\vec{a} \odot \vec{b} \odot \vec{c}$, $\odot = \langle , \rangle$ or "$\times$")

only very few combinations that make sense.

Eg. $\underbrace{(\vec{a} \cdot \vec{b})}_{\mathbb{R}} \times \vec{c}$ not defined.

but $\underbrace{\vec{a}}_{\mathbb{R}^3} \cdot (\underbrace{\vec{b} \times \vec{c}}_{\mathbb{R}^3}) \in \mathbb{R}$. makes sense.

$\vec{a} \times (\vec{b} \times \vec{c}) \in \mathbb{R}^3$ makes sense.

Prop: (i) $\vec{c} \cdot (\vec{a} \times \vec{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \det \begin{pmatrix} - \vec{c} - \\ - \vec{a} - \\ - \vec{b} - \end{pmatrix}$

(ii) $\vec{a} \times (\vec{b} \times \vec{c}) = \langle \vec{a}, \vec{c} \rangle \cdot \vec{b} - \langle \vec{a}, \vec{b} \rangle \cdot \vec{c}$

scalar multiplication.

Ex: Prove these identities!

Proof Geom. Prop. of $\vec{a}\times\vec{b}$ using (i)

(1): $\vec{a} \perp \vec{a}\times\vec{b}$ and $\vec{b} \perp \vec{a}\times\vec{b}$.

same

$$\langle \vec{a}, \vec{a}\times\vec{b} \rangle \overset{(i)}{=} \det\begin{pmatrix} - & \vec{a} & - \\ - & \vec{a} & - \\ - & \vec{b} & - \end{pmatrix} = 0 \Rightarrow \vec{a} \perp \vec{a}\times\vec{b}.$$

$$\langle \vec{a}, \vec{a}\times\vec{b} \rangle = \vec{a}\cdot(\vec{a}\times\vec{b})$$

Similarly for $\vec{b} \perp \vec{a}\times\vec{b}$

(2) & (3):

$$0 \leq \|\vec{a}\times\vec{b}\|^2 = \langle \vec{a}\times\vec{b}, \vec{a}\times\vec{b} \rangle \underset{\vec{c}=\vec{a}\times\vec{b}}{\overset{(i)}{=}} \det\begin{pmatrix} - & \vec{a}\times\vec{b} & - \\ - & \vec{a} & - \\ - & \vec{b} & - \end{pmatrix}$$

(2)

$$\Rightarrow \det\begin{pmatrix} - & \vec{a}\times\vec{b} & - \\ - & \vec{a} & - \\ - & \vec{b} & - \end{pmatrix} \geq 0 \Rightarrow \{\vec{a}\times\vec{b}, \vec{a}, \vec{b}\} \text{ ordered basis gives standard orientation.}$$

$$\Rightarrow \{\vec{a}, \vec{b}, \vec{a}\times\vec{b}\}$$

On the other hand,

$$\|\vec{a}\times\vec{b}\|^2 = \det\begin{pmatrix} - & \overset{\text{def}^{\underline{n}}}{\vec{a}\times\vec{b}} & - \\ - & \vec{a} & - \\ - & \vec{b} & - \end{pmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2$$

$$\overset{(Ex)}{=} \|\vec{a}\|^2\|\vec{b}\|^2 - \langle \vec{a}, \vec{b} \rangle^2$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2 \big(\underbrace{1 - \cos^2\theta}_{\sin^2\theta}\big)$$

Know:

$\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\|\|\vec{b}\|\cos\theta$

$\Rightarrow$ take sq root on both sides, get (3)

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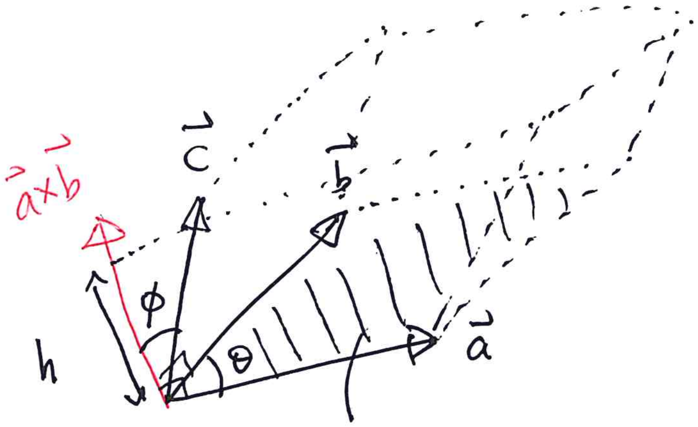
Geometric meaning of $\langle \vec{c}, \vec{a}\times\vec{b} \rangle$

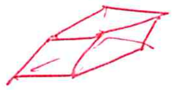


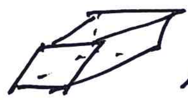
area $= \|\vec{a}\times\vec{b}\|$

$\langle \vec{c}, \vec{a}\times\vec{b} \rangle =$ (signed) volume of the $\underbrace{\text{parallelopiped}}_{\text{3D parallelogram}}$

$$\left| \vec{c}\cdot(\vec{a}\times\vec{b}) \right| = \|\vec{c}\|\,\|\vec{a}\times\vec{b}\| \cos\phi$$

$$= \underbrace{\left(\|\vec{c}\|\cos\phi\right)}_{\text{height of}}\ \underbrace{\left(\|\vec{a}\times\vec{b}\|\right)}_{\text{area of}}$$

height of    area of  "base area".

$= \pm \mathrm{vol}($  $)$.

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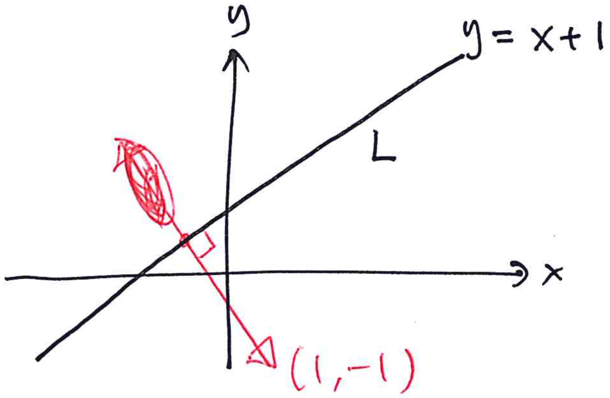
§ Lines and Planes in $\mathbb{R}^n$

1) Lines in $\mathbb{R}^2$



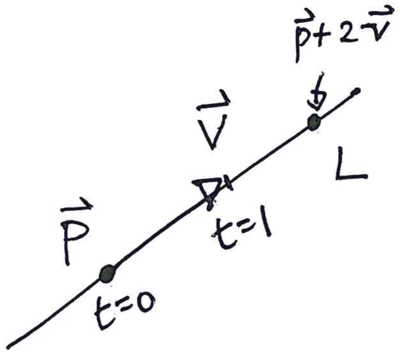
① Equation form

$$y = x + 1$$

or $\quad x - y = -1$

i.e. $\quad \underbrace{(1,-1)}_{\text{normal to } L}\cdot(x,y) = -1$

② Parametric Form (dynamic)



For a line L passing thr. a point $\vec{p}$ and parallel to $\vec{v}$:

$$L = \{ \vec{p} + t\vec{v} \mid t \in \mathbb{R} \}$$

free parameter.

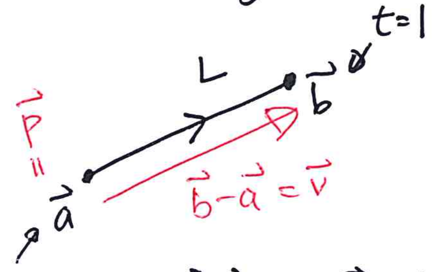
E.g. $\vec{p} = (1,2)$, $\vec{v} = (1,1)$

$$L = \{ (1,2) + t(1,1) \mid t \in \mathbb{R} \}$$

$$= \{ (\underbrace{1+t}_{x}, \underbrace{2+t}_{y}) \mid t \in \mathbb{R} \}$$

ie
$$(*) \begin{cases} x = 1+t \\ y = 2+t \end{cases}$$
parametric equations.

Line segments



$$L = \{ \vec{a} + t(\vec{b}-\vec{a}) \mid t \in [0,1] \}$$

③ Symmetric Form

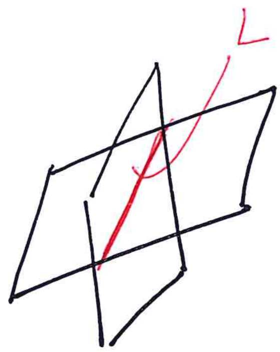
eliminate t from $(*)$.

$$\boxed{x-1 = y-2} \quad (=t)$$

2) Lines in $\mathbb{R}^3$

① Equation Form

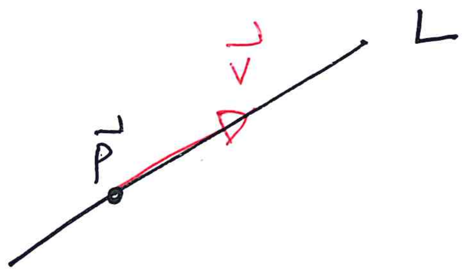
$$L : (\#) \begin{cases} x - y + z = 1 & \text{plane} \\ 2x + y - z = 0 & \text{plane} \end{cases}$$



2 planes (in general position (why?)) intersect along a line L $\Rightarrow$ 2 eq$^{\underline{n}}$ to describe a line in $\mathbb{R}^3$.

② Parametric Form

$$L = \{ \vec{p} + t\vec{v} \mid t \in \mathbb{R} \}$$



(same as before,
works for any n)

Q: Find a parametric form for the line L in (#)

Sol: "Solve equations".

$$(\#) \begin{cases} x - y + z = 1 & \text{———— (1)} \\ 2x + y - z = 0 & \text{———— (2)} \end{cases}$$

$(1) + (2) \Rightarrow \quad 3x = 1 \quad \Rightarrow \quad x = 1/3$

Sub. into (1) & (2)

$$\begin{cases} -y + z = 2/3 \\ y - z = -2/3 \end{cases} \text{ same.} \quad \Rightarrow \quad y = z - \frac{2}{3}$$

Take $z = t$ to be a parameter.

$$\Rightarrow \begin{cases} y = t - \frac{2}{3} \\ x = 1/3 \\ z = t \end{cases} \Leftrightarrow \begin{cases} x = 1/3 + 0 \cdot t \\ y = t - 2/3 \\ z = t + 0 \end{cases}$$ parametric eq^{n}.

$$L = \{ \underbrace{(\tfrac{1}{3}, -\tfrac{2}{3}, 0)}_{\vec{p}} + t \underbrace{(0,1,1)}_{\vec{v}} \mid t \in \mathbb{R} \}.$$

③ Symmetric form

E.g. Write $L = \{ (2,3,1) + t(-1,2,1) \mid t \in \mathbb{R} \}$ in symmetric form.

Sol:

$$\begin{cases} x = 2 - t \\ y = 3 + 2t \\ z = 1 + t \end{cases} \xrightarrow[t]{\text{eliminate}} t = \boxed{\frac{x-2}{-1} = \frac{y-3}{2} = \frac{z-1}{1}}$$

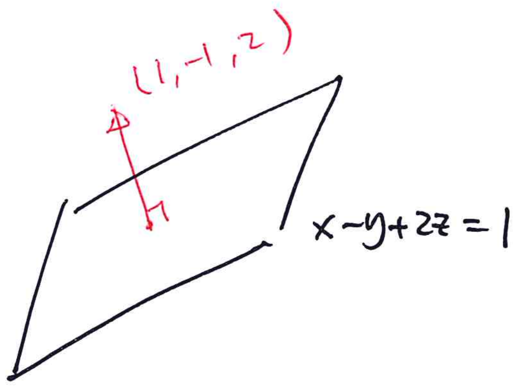
symmetric form.

3) Planes in $\mathbb{R}^3$

① Equation Form

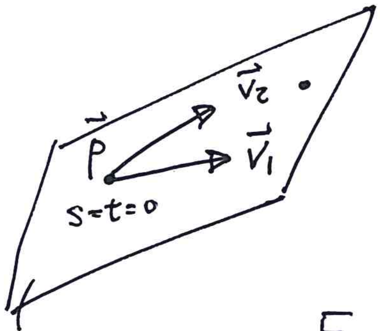
(**) — $\boxed{x - y + 2z = 1}$ plane P

i.e. $\underbrace{(1,-1,2)}_{\text{normal to } P} \cdot (x,y,z) = 1$



Ex: Show that the planes $x + y - z = 2$ is parallel to $x + y - z = 3$.

② Parametric Form (need 2 parameters to describe a plane
$\because$ There are 2 degrees of freedom)



$\mathbb{P}$

$$\mathbb{P} = \{ \vec{p} + \underbrace{t}_{} \vec{v_1} + \underbrace{s}_{} \vec{v_2} \mid t, s \in \mathbb{R} \}.$$

2 parameters

E.g. Represent $(**)$: $x - y + 2z = 1$
in parametric form.

Sol: "Solve eq$^{\underline{n}}$" only 1 eq$^{\underline{n}}$.

$$x - y + 2z = 1 \Rightarrow x = \underset{\substack{\parallel \\ t}}{y} - 2\underset{\substack{\parallel \\ s}}{z} + 1$$

$$\Rightarrow \begin{cases} x = t - 2s + 1 \\ y = t \\ z = s \end{cases}$$

parametric eq$^{\underline{n}}$.

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underset{\substack{\uparrow \\ \vec{p}}}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + t \underset{\substack{\uparrow \\ \vec{v_1}}}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} + s \underset{\substack{\uparrow \\ \vec{v_2}}}{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}.$$

$$\mathbb{P} = \{ (1,0,0) + t(1,1,0) + s(-2,0,1) \mid t, s \in \mathbb{R} \}.$$

※.